Anomaly Cancellation in Gauge Theories

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Abstract

We aim to establish a quantum analogue of Noether's theorem for minimally coupled gauge theories that exhibit an anomaly in the gauge symmetry by exploring the consequences of a mechanism of symmetry restoration. Using a path integral approach we show that the gauge anomaly has null expectation value in the vacuum, independently of the particularities of the model.

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I. INTRODUCTION

During the 80's, several theoretical evidences appeared to support the idea that anomalous gauge theories are not necessarily inconsistent. The work of Jackiw and Rajaraman [1], in which it was shown that a gauge anomalous two-dimensional theory can be well defined, was soon followed by the one of Faddeev and Shatashvilli [2], who noticed that the gauge anomaly could be canceled by the introduction of new quantum degrees of freedom, that transform second class constraints (correlated to the gauge anomaly) into first class ones. Such extra fields lead to a new effective action, a *gauge invariant* one. Then, it was understood independently by Harada and Tsutsui [3] and Babelon, Schaposnik and Viallet [4] that the application of Faddeev-Popov's method to an anomalous theory introduces these new degrees of freedom naturally, associated to the non-factorization of the integration over the gauge group. The arguments were no longer restricted to two dimensions.

It would be natural to consider what happens to the gauge anomaly in this new context, of gauge invariant effective actions. It is well known that there is a gauge anomaly in the intermediate theory (that obtained *after* integration over the fermion fields and *before* integration over the gauge fields). But would it survive this last integration? In this work we briefly review the approach mentioned above to the gauge anomaly through the use of functional methods, that incorporate in a natural way the extra degrees of freedom. Then, using this formalism, we show the cancellation of the gauge anomaly.

We organize the discussion as follows: in section II we review the arguments of Harada and Tsutsui that conduct to a gauge invariant formulation of an anomalous gauge theory and we show that the vacuum expectation value of the covariant divergence of the Noether current associated with global gauge symmetry has to vanish. In section III we present a very simple alternative derivation of the same results from another point of view, which is usually called gauge non-invariant formalism. We present our conclusions in section IV and in section V we give an appendix with an alternative derivation valid for the abelian case.

II. GAUGE INVARIANT FORMULATION OF ANOMALOUS GAUGE THEO-RIES

We briefly review the work of Harada and Tsutsui [3], which shows the way to restore gauge invariance on an anomalous gauge theory. We consider theories described by an action $I[\psi, \overline{\psi}, A_\mu]$, given by

$$I[\psi, \overline{\psi}, A_\mu] = I_G[A_\mu] + I_F[\psi, \overline{\psi}, A_\mu] \tag{1}$$
$$= \int dx \frac{1}{2} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + \int dx\, \overline{\psi} D \psi,$$

where dx indicates integration over a d-dimensional Minkowski space. The fields ψ are chiral fermions carrying the fundamental representation of $SU(N)$. As usual, A_μ takes values in the Lie algebra of $SU(N)$ such that

$$A_\mu = A_\mu^a T_a, \tag{2}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie\,[A_\mu, A_\nu],$$

and the generators T_a satisfy

$$[T_a, T_b] = i f_{abc} T_c, \qquad \mathrm{tr}\,(T_a T_b) = -\frac{1}{2} \delta_{ab}. \tag{3}$$

The operator D is the covariant derivative, and is called the Dirac operator of the theory. It is given by

$$D = i\gamma^\mu \left(\partial_\mu \mathbf{1} + ieA_\mu\right) \equiv i\gamma^\mu D_\mu. \tag{4}$$

Under gauge transformations,

$$g = \exp(i\theta^a(x)\, T_a), \tag{5}$$

and simultaneous changes of the fields ψ and A_μ as

$$A_\mu^g = g A_\mu g^{-1} + \frac{i}{e} \left(\partial_\mu g\right) g^{-1}, \tag{6}$$
$$\psi^g = g\psi,$$
$$\overline{\psi}^g = \overline{\psi} g^{-1},$$

the action I is classically gauge invariant

$$I[\psi^g, \overline{\psi}^g, A_\mu^g] = I[\psi, \overline{\psi}, A_\mu]. \tag{7}$$

Invariance of the action under *global* gauge transformations ($\partial_\mu \theta^a = 0$) leads to the classical covariant conservation of the current

$$(D_\mu)_{ab} J_b^\mu = 0, \tag{8}$$

with

$$J_a^\mu \equiv \overline{\psi} \gamma^\mu T_a \psi \tag{9}$$

and

$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} + e f_{abc} A_\mu^c. \tag{10}$$

The quantum theory is defined by the generating functional, which is

$$Z[\eta, \overline{\eta}, j_a^\mu] = \int d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu] + i \int dx [\overline{\eta}\psi + \overline{\psi}\eta + j_a^\mu A_\mu^a] \right). \tag{11}$$

It is well known [5] that, in the context of a non-anomalous theory, the integration over the field A_μ has to be restricted to configurations that are not physically equivalent, due to the gauge symmetry of the action. However, the non-invariance of the fermion measure under gauge transformations (characteristic of an anomalous theory) destroys gauge invariance and turns the Faddeev-Popov technique unnecessary. Moreover, it leads to a potential quantum violation of the classical conservation law (8). To see this, we perform the following infinitesimal change of variables

$$\psi^g = g\psi \approx (1 + i\delta\theta^a T_a)\psi, \tag{12}$$
$$\overline{\psi}^g = \overline{\psi} g^{-1} \approx \overline{\psi}(1 - i\delta\theta^a T_a).$$

Imposing that the result of the integral should be the same for both variables, we have

$$Z = Z_g. \tag{13}$$

In this way

$$Z_g = \int d\psi^g d\overline{\psi}^g dA_\mu \exp\left(iI[\psi^g, \overline{\psi}^g, A_\mu] + i \int dx \left[\overline{\eta}\psi^g + \overline{\psi}^g \eta + j_a^\mu A_\mu^a \right] \right) \tag{14}$$
$$= \int J[A_\mu, g] d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu] + i \int dx \left[\overline{\eta}\psi + \overline{\psi}\eta + j_a^\mu A_\mu^a \right] \right.$$
$$\left. + i \int dx \left(i\delta\theta^a \left[i (D_\mu)_{ab} J_b^\mu + \overline{\eta} T_a \psi - \overline{\psi} T_a \eta \right] \right) \right).$$

We notice the appearance of a Jacobian $J[A_\mu, g] = J[A_\mu, \delta\theta]$. Given the infinitesimal character of the transformation, it can be functionally expanded to first order in $\delta\theta$

$$J[A_\mu, \delta\theta] = 1 + \int dx \, \delta\theta^a \mathcal{A}_a(A_\mu) + ..., \tag{15}$$

and imposing $Z = Z_g$, we obtain

$$= \int d\psi d\overline{\psi} dA_\mu \int dx \, i\delta\theta^a [i \, (D_\mu)_{ab} \, J_b^\mu - i\mathcal{A}_a(A_\mu) + \overline{\eta} T_a \psi - \overline{\psi} T_a \eta] \tag{16}$$

$$\times \exp\left(iI[\psi, \overline{\psi}, A_\mu] + i \int dx \left[\overline{\eta}\psi + \eta\overline{\psi} + j_a^\mu A_\mu^a\right]\right)$$

$$= 0.$$

Then, setting the external sources to zero, we see that

$$\int d\psi d\overline{\psi} dA_\mu \left\{(D_\mu)_{ab} \, J_b^\mu\right\} \exp(iI[\psi, \overline{\psi}, A_\mu]) \tag{17}$$

$$= \int d\psi d\overline{\psi} dA_\mu \left\{\mathcal{A}_a(A_\mu)\right\} \exp(iI[\psi, \overline{\psi}, A_\mu]),$$

which gives us

$$\left\langle 0 \left|(D_\mu)_{ab} \, J_b^\mu\right| 0\right\rangle = \left\langle 0 \left|\mathcal{A}_a(A_\mu)\right| 0\right\rangle. \tag{18}$$

So, from the functional integral point of view, it has long been clear [6] that the possible anomaly in the gauge symmetry is intrinsically related to the non-invariance of the fermionic measure. However, we notice that there is still an expectation value to be taken, before we definitely say that current conservation is violated.

Coming back to equation (11) we notice that, if we proceed applying Faddeev-Popov's method[1], the gauge volume does not factor out, since there is an additional dependence on the group elements coming from the Jacobian,

$$d\psi^g d\overline{\psi}^g = J[A_\mu, g] d\psi d\overline{\psi} \equiv \exp(i\alpha_1[A_\mu, g]) d\psi d\overline{\psi}, \tag{19}$$

where we introduced the *Wess-Zumino functional* $\alpha_1[A_\mu, g]$. Introducing the famous "1" of Faddeev-Popov,

$$1 = \Delta_{\text{FP}}[A_\mu] \int dg \, \delta\left(f\left[A_\mu^g\right]\right) \tag{20}$$

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[1] This is not mandatory, as will be explained in the next section.

in the vacuum amplitude $Z[0]$ (with $\Delta_{\text{FP}}[A_\mu]$ being the Faddeev-Popov determinant and $f(A_\mu) = 0$ being the gauge fixing condition) we see that

$$
\begin{aligned}
Z[0] &= \int d\psi d\bar{\psi} dA_\mu dg\, \Delta_{\text{FP}}[A_\mu]\, \delta\left(f\left[A_\mu^g\right]\right) \exp\left(iI\left[\psi, \bar{\psi}, A_\mu\right]\right) \qquad (21) \\
&= \int d\psi d\bar{\psi} dA_\mu^{g^{-1}} dg\, \Delta_{\text{FP}}\left[A_\mu^{g^{-1}}\right] \delta\left(f\left[\left(A_\mu^{g^{-1}}\right)^g\right]\right) \exp\left(iI\left[\psi, \bar{\psi}, A_\mu^{g^{-1}}\right]\right) \\
&= \int d\psi d\bar{\psi} dA_\mu dg\, \Delta_{\text{FP}}[A_\mu]\, \delta\left(f[A_\mu]\right) \exp\left(iI\left[\psi^g, \bar{\psi}^g, A_\mu\right]\right) \\
&= \int d\psi d\bar{\psi} dA_\mu dg\, \Delta_{\text{FP}}[A_\mu]\, \delta\left(f[A_\mu]\right) \exp\left(iI\left[\psi, \bar{\psi}, A_\mu\right] + i\alpha_1\left[A_\mu, g^{-1}\right]\right).
\end{aligned}
$$

The non-fatorization of the gauge volume naturally generates new degrees of freedom, the *Wess-Zumino fields* $\theta^a(x)$, that come from the integration over the local group element $g(x)$

$$
dg = \prod_a d\theta_a(x). \qquad (22)
$$

Following the spirit of [3], we show below that these degrees of freedom produce a new gauge invariant action. To see this, we define

$$
\exp(iW[A_\mu]) := \int d\psi d\overline{\psi} \exp\left(iI\left[\psi, \bar{\psi}, A_\mu\right]\right). \qquad (23)
$$

The Jacobian can be related to $W[A_\mu]$ in the following way[2]:

$$
\begin{aligned}
\exp(iW[A_\mu^g]) &= \int d\psi d\overline{\psi} \exp\left(i\int dx\, \overline{\psi} D(A_\mu^g)\psi\right) \qquad (24) \\
&= \int d\psi d\overline{\psi} \exp\left(i\int dx\, \overline{\psi}^{g^{-1}} D(A_\mu)\psi^{g^{-1}}\right) \\
&= J[A_\mu, g]\exp(iW[A_\mu]) \\
&\rightarrow J[A_\mu, g] = \exp\left(i\left[W[A_\mu^g] - W[A_\mu]\right]\right).
\end{aligned}
$$

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[2] We observe that $\alpha_1[A_\mu, g] = -\alpha_1[A_\mu, g^{-1}]$. This can be easily shown using the definition of the Jacobian

$$
d\psi^g d\overline{\psi}^g = \exp\left(i\alpha_1[A_\mu, g]\right) d\psi d\overline{\psi}
$$

to write the formula

$$
\begin{aligned}
d\left(\psi^g\right)^{g^{-1}} d\left(\overline{\psi}^g\right)^{g^{-1}} &\equiv d\psi d\overline{\psi} \\
&= \exp\left(i\alpha_1[A_\mu, g^{-1}]\right) d\psi^g d\overline{\psi}^g.
\end{aligned}
$$

So, we see that α_1 is given by

$$\alpha_1[A_\mu, g] = W[A_\mu^g] - W[A_\mu], \tag{25}$$

and exhibits clearly its behavior under gauge transformations

$$\alpha_1[A_\mu^h, g] = W[A_\mu^{hg}] - W[A_\mu^h] \tag{26}$$
$$= \alpha_1[A_\mu, hg] - \alpha_1[A_\mu, h].$$

Now we define an effective action integrating over the fermions and Wess-Zumino fields:

$$\exp(iI_{\text{eff}}[A_\mu]) := \int d\psi d\overline{\psi} dg \exp(iI[\psi, \overline{\psi}, A_\mu] + i\alpha_1[A_\mu, g^{-1}]) \tag{27}$$
$$= \int d\psi d\overline{\psi} dg \exp(iI[\psi, \overline{\psi}, A_\mu] + iW[A_\mu^{g^{-1}}] - iW[A_\mu])$$
$$= \int dg \exp(iW[A_\mu^{g^{-1}}]).$$

This new action is gauge invariant, as is shown below:

$$\exp(iI_{\text{eff}}[A_\mu^h]) = \int dg \exp(iW[(A_\mu^h)^{g^{-1}}]) = \int dg \exp(iW[A_\mu^{(gh^{-1})^{-1}}]) \tag{28}$$
$$= \int d\left(gh^{-1}\right) \exp(iW[A_\mu^{(gh^{-1})^{-1}}]) = \exp(iI_{\text{eff}}[A_\mu])$$

This gauge invariance of the effective action strongly indicates the cancellation of the anomaly, as long as gauge symmetry is restored at quantum level. To investigate this, we first review the steps that conduct to a familiar expression [7] of the anomaly in terms of $W[A_\mu]$:

$$\mathcal{A}_a(x) = i \left.\frac{\delta\alpha_1[A_\mu, \theta]}{\delta\theta_a(x)}\right|_{\theta=0} = i \left.\frac{\delta W[A_\mu^\theta]}{\delta\theta_a(x)}\right|_{\theta=0} \tag{29}$$
$$= i \int dz \left.\frac{\delta W[A_\mu^\theta]}{\delta A_{\mu,b}^\theta(z)} \frac{\delta A_{\mu,b}^\theta(z)}{\delta\theta_a(x)}\right|_{\theta=0}$$
$$= i \int dz \frac{\delta W[A_\mu]}{\delta A_{\mu,b}(z)} \left(\left.\frac{\delta A_{\mu,b}^\theta(z)}{\delta\theta_a(x)}\right|_{\theta=0}\right).$$

Using that

$$\left.\frac{\delta A_{\mu,b}^\theta(z)}{\delta\theta_a(x)}\right|_{\theta=0} = \frac{1}{e}(D_\mu)_{ab}\,\delta(z-x) \tag{30}$$

we obtain

$$\mathcal{A}_a(x) = -\frac{i}{e}\left(D_\mu \frac{\delta W[A_\mu]}{\delta A_\mu(z)}\right)_a. \tag{31}$$

Now, we consider a gauge transformation of the infinitesimally gauge transformed $A_\mu^{\delta\theta}$,

$$\left(A_\mu^{\delta\theta}\right)_a^{g^{-1}} \approx \left(A_\mu^{g^{-1}}\right)_a + \frac{1}{e}\left(g^{-1}\left(D_\mu\delta\theta\right)g\right)_a \tag{32}$$
$$= \left(A_\mu^{g^{-1}}\right)_a + \frac{1}{e}\left(D_\mu^{g^{-1}}\right)_b^a\left(\delta\theta^{g^{-1}}\right)^b,$$

with

$$\delta\theta^{g^{-1}} = g^{-1}\delta\theta g. \tag{33}$$

Then we see that

$$\exp(iI_{\text{eff}}\left[A_\mu^{\delta\theta}\right]) = \int dg \exp\left(iW[A_\mu^{g^{-1}} + \frac{1}{e}D_\mu^{g^{-1}}\delta\theta^{g^{-1}}]\right) \tag{34}$$
$$= \int dg \exp\left(iW[A_\mu^{g^{-1}}]\right)\left(1 + \frac{i}{e}\int dx\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}a}(x)}\left(D_\mu^{g^{-1}}\delta\theta^{g^{-1}}\right)_a(x)\right)$$
$$= \exp\left(iI_{\text{eff}}\left[A\right]\right)$$
$$- \frac{i}{e}\int dx \int dg \left(\delta\theta^{g^{-1}}\right)^a(x)\left[D_\mu^{g^{-1}}\left(\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(x)}\right)\right]_a \exp\left(iW[A_\mu^{g^{-1}}]\right).$$

Remembering the gauge invariance of I_{eff}, we conclude that

$$\int dx \int dg \left(\delta\theta^{g^{-1}}\right)^a(x)\left[D_\mu^{g^{-1}}\left(\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(x)}\right)\right]_a \exp\left(iW[A_\mu^{g^{-1}}]\right) = 0. \tag{35}$$

Integrating over A_μ, changing variables $A_\mu \to A_\mu^g$ and using the gauge invariance of the bosonic measure $(dA_\mu^g = dA_\mu)$ we get

$$\int dg \left(\delta\theta^{g^{-1}}\right)^a(x)\int dA_\mu \left[D_\mu\left(\frac{\delta W[A_\mu]}{\delta A_\mu(x)}\right)\right]_a \exp\left(iW[A_\mu]\right) = 0. \tag{36}$$

Finally, taking into account the arbitrariness of $\delta\theta$, we obtain the cancellation of the anomaly:

$$0 = \int dA_\mu \left[D_\mu\left(\frac{\delta W[A_\mu]}{\delta A_\mu(x)}\right)\right]_a \exp\left(iW[A]\right) \tag{37}$$
$$\Rightarrow \langle 0\left|\mathcal{A}_a(A_\mu)\right|0\rangle = 0.$$

This result confirms the main expectation concerning gauge invariance of the effective action: the current is covariantly conserved *at the full quantum level*

$$\langle 0\left|(D_\mu)_{ab} J_b^\mu\right|0\rangle = \langle 0\left|\mathcal{A}_a(A_\mu)\right|0\rangle = 0. \tag{38}$$

III. GAUGE NON-INVARIANT FORMULATION OF ANOMALOUS GAUGE THEORIES

In the previous section we emphasized the fact that the anomaly has a null expectation value in the vacuum

$$\langle 0 \left| \mathcal{A}_a(A_\mu) \right| 0 \rangle = 0. \tag{39}$$

We could reach equivalent conclusions independently for the vacuum expectation value of the covariant divergence of the current. To see this, we simply consider a bosonic change of variables in the functional integral:

$$Z[0] = \int d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) = \int d\psi d\overline{\psi} dA_\mu^g \exp\left(iI[\psi, \overline{\psi}, A_\mu^g]\right) \tag{40}$$

$$= \int d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu^g]\right),$$

where we used again the invariance of the bosonic measure $dA_\mu^g = dA_\mu$. The functional integral does not contain a gauge group volume, as it would happen in a non-anomalous gauge theory, because fermion integration produces a gauge non-invariant $W[A_\mu]$. So, the Faddeev- Popov trick is unnecessary here, as it has already been stressed. This way of facing the problem is known as *gauge non-invariant representation*[8].

Next, we consider an infinitesimal gauge transformation, characterized by $g \approx 1 + i\delta\theta^a T_a$,

$$I[\psi, \overline{\psi}, A_\mu^g] = I[\psi, \overline{\psi}, A_\mu + \frac{1}{e} D_\mu \delta\theta] \tag{41}$$

$$= I[\psi, \overline{\psi}, A_\mu] + \frac{1}{e} \int dx \, (D_\mu \delta\theta(x))_a \frac{\delta I}{\delta A_\mu^a(x)}$$

$$= I[\psi, \overline{\psi}, A_\mu] - \frac{1}{e} \int dx \, \delta\theta^a(x) \left(D_\mu \frac{\delta I}{\delta A_\mu(x)} \right)_a.$$

This gives

$$Z[0] = \int d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) \tag{42}$$

$$\approx \int d\psi d\overline{\psi} dA_\mu \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right)$$

$$- \frac{1}{e} \int dx \, \delta\theta^a(x) \int d\psi d\overline{\psi} dA_\mu \left(D_\mu \frac{\delta I}{\delta A_\mu(x)} \right)_a \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right)$$

$$\Rightarrow \int d\psi d\overline{\psi} dA_\mu \left(D_\mu \frac{\delta I}{\delta A_\mu(x)} \right)_a \exp\left(iI[\psi, \overline{\psi}, A_\mu]\right) = 0.$$

Remembering that

$$\frac{\delta I}{\delta A_\nu^a(x)} = (D_\mu F^{\mu\nu})_a - e\overline{\psi}\gamma^\nu T_a \psi, \tag{43}$$

and that $(D_\mu D_\nu F^{\mu\nu})_a = 0$ identically,

$$\left(D_\mu \frac{\delta I}{\delta A_\mu(x)}\right)_a = (D_\mu)_{ab}\,\overline{\psi}\gamma^\mu T_b \psi, \tag{44}$$

we thus showed that

$$\left\langle 0 \left| (D_\mu)_{ab}\,\overline{\psi}\gamma^\mu T_b \psi \right| 0 \right\rangle = \left\langle 0 \left| (D_\mu)_{ab}\, J_b^\mu \right| 0 \right\rangle = 0. \tag{45}$$

We notice that this result was reached without making any fermionic change of variables. The above equation is completely consistent with our previous conclusions.

IV. CONCLUSION

The functional formalism points out the origin of the gauge anomaly and a road for its formal cancellation. Restoration of gauge symmetry implies a null expectation value for the anomaly. This cancellation suggests that anomalies are not an obstacle to the quantization of theories involving chiral fermions. The usual argument is that anomalies destroy Slavnov-Taylor identities, necessary to relate renormalization constants and prove the renormalizability of the theory. On the basis of our results, there is no reason to believe that Slavnov-Taylor identities are not preserved in a gauge anomalous theory. A detailed analysis of the perturbative renormalization procedure under this new perspective would be very important and will be considered in detail in the future.

V. APPENDIX

In the abelian case, there is an alternative approach to the result obtained in the gauge invariant representation. First we notice that the gauge transform of the anomaly can be

written as a functional derivative

$$\mathcal{A}^\theta = \left(-\frac{i}{e}\partial_\mu \frac{\delta W\left[A_\mu\right]}{\delta A_\mu\left(z\right)}\right)^\theta = -\frac{i}{e}\partial_\mu \frac{\delta W\left[A_\mu^\theta\right]}{\delta A_\mu^\theta\left(x\right)} \tag{46}$$

$$= \frac{i}{e}\int dz \frac{\delta W\left[A_\mu^\theta\right]}{\delta A_\mu^\theta\left(z\right)}\partial_\mu\delta\left(z-x\right)$$

$$= i\int dz \frac{\delta W\left[A_\mu^\theta\right]}{\delta A_\mu^\theta\left(z\right)}\frac{\delta A_\mu^\theta\left(z\right)}{\delta\theta\left(x\right)}$$

$$= i\frac{\delta W\left[A_\mu^\theta\right]}{\delta\theta\left(x\right)} = i\frac{\delta\alpha_1\left[A_\mu,\theta\right]}{\delta\theta\left(x\right)}.$$

Now, it is easy to show that the vacuum expectation value for the anomaly can be computed as

$$\langle\mathcal{A}\left(x\right)\rangle = i\int d\psi d\overline{\psi}dA_\mu\left(\mathcal{A}\left(x\right)\right)\exp(iI\left[\psi,\overline{\psi},A_\mu\right]) \tag{47}$$

$$= i\int d\psi d\overline{\psi}dA_\mu d\theta\Delta_{\text{FP}}[A_\mu]\delta(f(A_\mu^\theta))\left(\mathcal{A}\left(x\right)\right)\exp\left(iI\left[\psi,\overline{\psi},A_\mu\right]\right)$$

$$= i\int d\psi d\overline{\psi}dA_\mu d\theta\Delta_{\text{FP}}[A_\mu]\delta(f(A_\mu))\left(\mathcal{A}^{-\theta}\left(x\right)\right)\exp(iI\left[\psi,\overline{\psi},A_\mu\right]+i\alpha_1\left[A_\mu,-\theta\right]),$$

where we performed the usual steps of the Faddeev-Popov method but took into consideration that the fermionic measure is not invariant under a gauge transformation. Definig $\mathcal{D}A_\mu$ as

$$\mathcal{D}A_\mu := dA_\mu\Delta_{\text{FP}}[A_\mu]\delta(f(A_\mu)), \tag{48}$$

we can proceed, using the result just derived

$$\langle\mathcal{A}\left(x\right)\rangle = i\int d\psi d\overline{\psi}\mathcal{D}A_\mu d\theta\left(-\frac{\delta}{\delta\theta\left(x\right)}\alpha_1\left[A_\mu,-\theta\right]\right) \tag{49}$$

$$\times\exp(iI\left[\psi,\overline{\psi},A_\mu\right]+i\alpha_1\left[A_\mu,-\theta\right])$$

$$= -\int d\psi d\overline{\psi}\mathcal{D}A_\mu d\theta\frac{\delta}{\delta\theta\left(x\right)}[\exp(iI\left[\psi,\overline{\psi},A_\mu\right]+i\alpha_1\left[A_\mu,-\theta\right])]$$

$$= -\int d\theta\frac{\delta}{\delta\theta\left(x\right)}F\left[\theta\right] = 0.$$

which shows that the anomaly vanishes because of the translational invariance of the funcional measure [9]. The same approach is not possible to be applied to the non-abelian case, since it is not possible to prove that the gauge transform of the anomaly is a functional derivative with respect to the Wess-Zumino fields [10].

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[8] See section 14.3 of Abdalla, E., Abdalla, M. C. B and Rothe, K., *Non-Perturbative Methods in Two-Dimensional Quantum Field Theory*, 2$^\text{nd}$ edition, World Scientific, Singapore, 2001; a detailed analysis of gauge invariant and gauge non-invariant Green's functions for a specific anomalous theory (the chiral Schwinger model) in both formalisms can be found in C. A. Linhares, H. J. Rothe and K. D. Rothe, *Phys. Rev.* **D35** (1987) 2501-2509.

[9] The argument that functional integrals of functional derivatives have to vanish due to translation invariance of the measure can be found, for example, in the fourth chapter of the book of R. J. Rivers, *Path Integral Methods in Quantum Field Theory,* Cambridge Monographs on Mathematical Physics, Cambridge University Press, USA, 1988.

[10] Rafael Chaves Souto Araújo, *Anomaly Cancellation in Non-Abelian Gauge Theories*, Master thesis (in portuguese), CBPF, Rio de Janeiro, 2006.